Hamburger Mathematics: An Application of Newton’s Law of Cooling

Dr. Frank’s Math Minute:
Dr. Frank's Math Minutes are short and fun video clips that show the power and applicability of mathematics. They are accompanied by free worksheets designed for students with varying levels of mathematics knowledge. The video to accompany this worksheet can be found at:

http://www.wangeducation.com/mathminute/hamburger.shtml

If you found this lesson useful, you might also be interested in other Wang Education products. Wang Education produces a growing line of mathematics enrichment products for students as well as classroom management products for teachers.

Visit us at:

http://www.wangeducation.com

Prerequisites:
High school algebra through a study of logarithms (preferably through a study of natural logarithms although knowledge through just common logarithms is fine. A brief discussion of *e* is given in this handout.)

Students who have taken 2 years of algebra should be able to understand this handout, conduct the experiment, collect data, graph the data from the experiment, determine the equation of the function that theoretically models the temperature of the object given by Newton’s Law of Cooling, and then graph this function.

Background:
This worksheet is designed to be used in conjunction with Dr. Frank’s Math Minute “Hamburger Mathematics – An Application of Newton’s Law of Cooling.” On the video, Dr. Frank records the temperature of a hamburger after it leaves the grill over a period of 25 minutes. At 1 minute intervals, he plots the temperature on a graph. The graph displays temperature marked along the vertical axis and time marked along the horizontal axis. Over time, the hamburger begins to cool.

What equation can describe the graph showing the temperature of the hamburger over time? Fully understanding the equation that describes the cooling of the hamburger requires students to know and understand how to solve a differential equation. Typically, differential equations are not taught until the end of a calculus course or as a separate course after calculus. In this worksheet, we will not attempt to teach the mechanics of solving differential equations. We will, however, try to give a flavor for what calculus and differential equations are all about. A basic knowledge of high school algebra through the study of logarithms and a little bit of patience and concentration are all that is needed to work through this worksheet.

Introduction:
Once a burger leaves the grill it begins to cool. The cooling process of the burger can be described by Newton’s Law of Cooling. Newton, considered to be one of the greatest mathematicians and scientists ever to have lived, developed and published his law anonymously in 1701.

Newton’s Law of Cooling basically says that the rate at which an object cools is proportional to the difference of the temperature of the object and the surrounding air. In other words, an object cools faster if it is much hotter than the surrounding air. This intuitively makes sense. If you take a hot cup of water and place in on the countertop, it will cool pretty fast at the beginning but then after a period of time it will cool more slowly. After some time, it may even seem as if the water is not cooling at all as it is cooling so slowly as not to be noticeable.

What we describe in words above can be described mathematically below in what we call a differential equation. A differential equation is an equation that involves functions and the derivatives of functions. Derivatives themselves are functions that express the rate of change of another function.

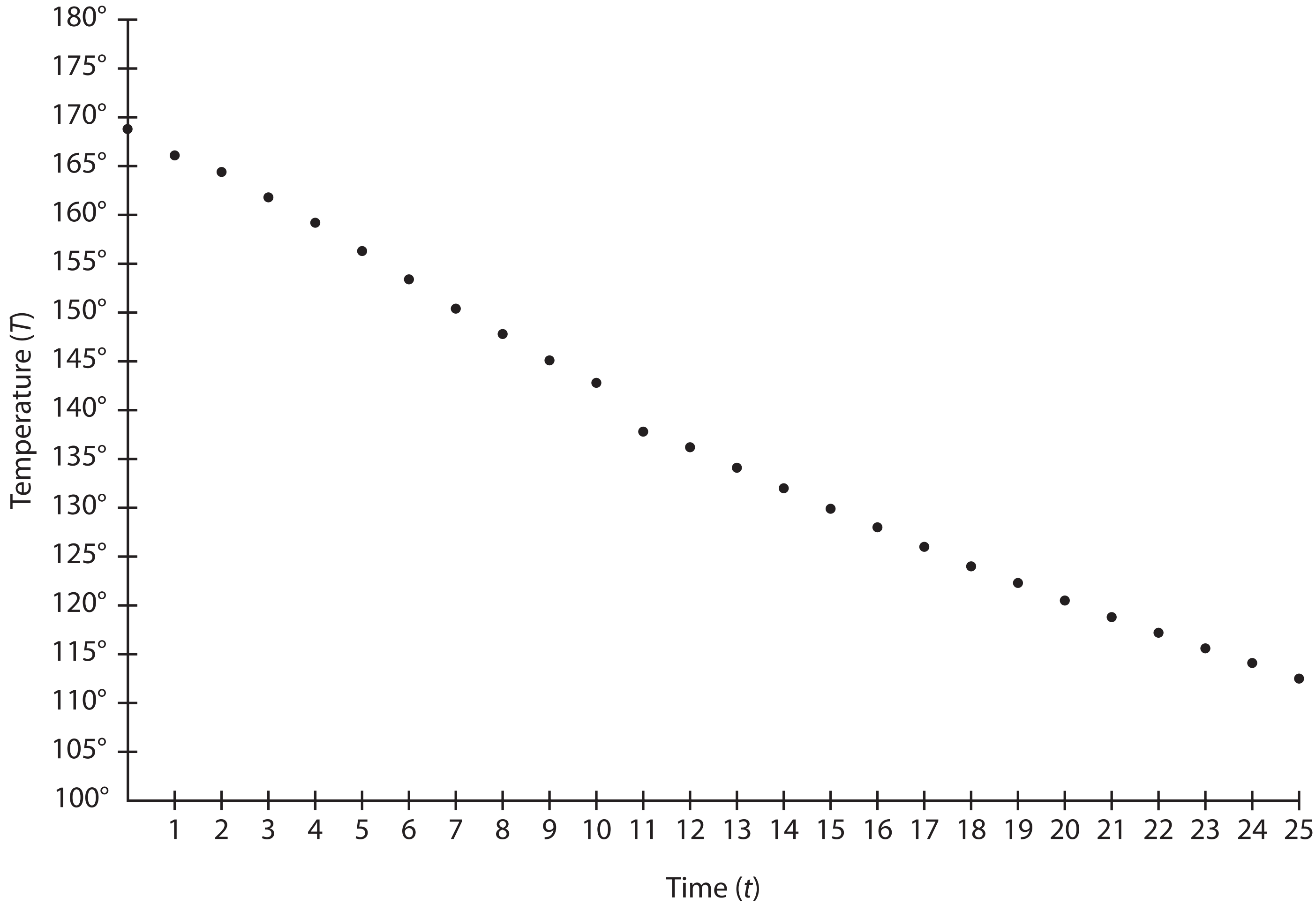
One of the central ideas of calculus is that of "rate of change". If a letter, f, is used to name a function, then we use the letter with a little tick mark to the upper right, f', to designate the derivative function of f. The derivative function f', read "f prime," is itself a function that describes the rate of change of the function f. In geometric terms, f' is the function that describes the slope of the graph of f as a function of the independent variable. Don't worry if you don't fully understand these details. The concept of derivatives and how they are computed will be taught in a calculus course.

In the experiment shown on *Dr. Frank's Math Minute*, T is used to represent the temperature function. T is a function of t, which represents time. This can be written as $T(t)$. Little t is measured in minutes beginning with 0, the time when the first reading is taken. $T(0)$ is called the initial temperature of the hamburger because it is the temperature of the hamburger at time $t = 0$. In our case, $T(0) = 168.8$ degrees Fahrenheit. At 1 minute after this first reading is taken, the temperature of the hamburger is 166.1° F. Therefore, we would write $T(1) =$ 166.1°F.

We can construct a chart of the temperature and the time for the experiment in *Dr. Frank's Math Minute* which we show here

Time (t)	Temperature (T)	Time (t)	Temperature (T)
0	168.8	13	134.1
1	166.1	14	132.0
2	164.4	15	129.9
3	161.8	16	128.0
4	159.2	17	126.0
5	156.3	18	124.0
6	153.4	19	122.3
7	150.4	20	120.5
8	147.8	21	118.8
9	145.1	22	117.2
10	142.8	23	115.6
11	137.8	24	114.7
12	136.2	25	112.5

We can graph these data points on a graph with time measured along the horizontal axis and temperature along the vertical axis as we did on the video.



Now, our goal is to describe an equation for T that will allow us to predict what the temperature is at a given time.

We begin with a mathematical expression for Newton's Law of Cooling.

The differential equation that describes Newton's Law of Cooling is:

$$T'(t) = -K[T(t) - T_A] \qquad \textbf{Equation (1)}$$

where

> T is the temperature function. It describes the temperature of the burger as a function of time t. In this experiment, t is measured in minutes beginning with $t = 0$ being the time when the first measurement is taken.
>
> T' is the derivative of the temperature function. T' is itself a function that tells the rate of change of the function T at various times. For example, if $T'(2 \text{ minutes}) = -2° \text{ F},$ this means that 2 minutes after the initial (or first) reading, the rate of change of the temperature at that point in time would be –2 degrees per minute.
>
> $T(0)$ is the initial temperature of the burger. This is the temperature of the burger at time $t = 0,$ the time we begin taking measurements. In our experiment, $T(0) = 168.8.$ For example, at $t = 2$ minutes, the temperature is 164.4 degrees F so $T(2) = 164.4.$

T_A is the ambient temperature, meaning it is the temperature of the air surrounding the hamburger. (In our experiment, T_A was equal to 71 degrees Fahrenheit.)

Note that the function T is buried in Equation (1). Mathematicians say that T is defined *implicitly*. As students, you are likely used to working with functions that are described *explicitly*. For example, in the equation $f(x) = x^2$ the function f is described explicitly. In comparison, in the equation $x^2 + [g(x)]^2 = 9$, the function g is defined implicitly. The advantage of having a function described explicitly is that it is easy to compute values of the function for various values of the independent variable (here, the independent variable is x). Therefore, it is easy to see that $f(2) = 4$ however, it is not that easy to see that $g(2) = \pm\sqrt{5}$.

When presented with a differential equation, the challenge is to solve for the function. Here, the function for which we want to find the equation is the temperature function. We have an equation involving the function T as well as the function T'. When we say that we want to *solve a differential equation*, we want to find the equation of the function or functions in the differential equation.

Solving a differential equation involves applying knowledge of functions and their derivatives. We won't get into the specifics of how to solve the differential equation, but we will give its solution. The solution of the differential equation above is:

$$T(t) = T_A + (T_0 - T_A)e^{-Kt}$$

where

T_A = temperature of the surrounding air

T_0 = temperature of the burger at $t = 0$, the time that measurements began

Here, e is a special number that has a value about equal to 2.7128. It is a number like π whose decimal approximation goes on forever. Like π, it is irrational and so cannot be precisely described by a fraction of two whole numbers. Here e is used as the base of an exponential expression in the same way that the number 10 or 2 may be used as the base of an exponential expression. It turns out that e arises quite frequently in the study of mathematics just as π. Though π can be described as the ratio of the circumference of a circle to its diameter, there really is not such a simple explanation for e. [One way e can be described is that it arises in the study of the compounding of money. A bank can compound interest annually, monthly, weekly, etc. As a bank increases the frequency at which it compounds (i.e., weekly, daily, hourly, etc.) there comes a limiting point to the interest generated. This limiting point occurs when the compounding is *continuous*. It is in the calculation of this limiting point that the number e arises.]

We plug in the information we already have into the equation for T.

$$T(t) = 71° + (168.8° - 71°)e^{-Kt}$$

We simplify to get:

$$T(t) = 71° + (97.8°)e^{-Kt} \qquad \textbf{Equation (2)}$$

We are not quite done yet as we still need to determine the value of K. The letter K in the equation is a constant (meaning, a particular number). To find K, we take the information about one of the data points (other than the first data point), plug the information into the equation for T and t and then solve for K. We will use the data point at t = 25 minutes. The temperature of the hamburger is 112.5° at $t = 25$ minutes. Plugging this into the equation, we get:

$$T(25) = 71° + (97.8°)e^{-K25} \qquad \textbf{Equation (3)}$$

$$112.5° = 71° + (97.8°)e^{-K25}$$

Subtracting 71° from both sides and rewriting $K25$ as $25K$, we get:

$$41.5° = 97.8°e^{-25K} \qquad \textbf{Equation (4)}$$

Now, we divide both sides by 97.8° (using a calculator) to get:

$$.4243354 = e^{-25K}$$

.4243354 – the number on the left of the equation – is a decimal approximation of $\frac{41.5}{97.8}$.

The next step requires a bit of knowledge of logarithms. We now take the natural logarithm of both sides of the equation. A natural logarithm is a logarithm whose base is *e*. (Remember that a common logarithm is a logarithm whose base is 10.) The symbol for the natural logarithm function is a lower case L followed by a lowercase N: ln.

Now we take the natural log of both sides (using a calculator) and then apply the properties of logarithms to simplify.

$$\ln .4243354 = \ln e^{-25K} \qquad \textbf{Equation (5)}$$

$$-.857231 = -25K \qquad \textbf{Equation (6)}$$

Solving for *K*, we get:

$$K = .034289 \qquad \textbf{Equation (7)}$$

Now, finally, we have an explicit equation for the temperature function for our experiment. It is:

$$T(t) = 71° + (97.8°)e^{-.034289t} \qquad \textbf{Equation (8)}$$

You can now go to the graphing calculator or a computer and use any math graphing program to graph this equation. Visually check how well the graph of the *T* given by Newton's Law of Cooling fits the data from the experiment. In *Dr. Frank's Math Minute,* we used a special curve fitting program to draw a "best fit" line through our data points and then compare this curve to the curve predicted by Newton's Law of Cooling; however, in your experiment, we will not ask you to find the "best fit" line to your data but merely visually check how well the data points fit the curve predicted by Newton's Law of Cooling. If your experiment goes well, the data collected from the experiment should be a close fit for the graph of the equation that arises from Newton's Law of Cooling.

Experiment

Now, conduct the experiment yourself. You need not use a hamburger. Anything that is heated and set out to cool would work fine, such as a cup of warm water. You will also need a watch or a stopwatch. We will assume in this experiment that you are using a stopwatch. If you are using a watch, you will need to make some adjustments in what I am describing so that you consider a particular time as the starting time 0 on the stopwatch.

a. Use the graph paper that is attached and label the vertical axis "temperature" and then label the horizontal axis "time". Choose a scale that will allow you to graph the range of temperatures over the time you desire and label the time and temperature increments. (Our suggestion is to measure temperature at 1 minute intervals over 25 minutes.)

b. Next, find the equation that describes the temperature of the object over time as it cools. Use the equation:

$$T(t) = T_A + (T_0 - T_A)e^{-Kt}$$

Replace T_A in the equation with the temperature of the surrounding air. Replace T_0 with the temperature of the object when you start taking measurements.

Then, simplify the expression you have so that the equation looks like **Equation (2).**

c. The next step is to solve for the constant *K*. This can be accomplished by taking one of the temperature readings. Take the temperature of the object at any given time *t* after the measurements have begun. Plug this information into the equation as shown in **Equation (3).**

Apply arithmetic to simplify the expression just found so that it looks like **Equation (4).**

Take the natural logarithm of both sides so that the resulting equation looks like **Equation (5).**

Solve for *K* as shown in **Equations (6)** and **(7).**

Finally, express the equation of the Temperature function as in **Equation (8).**

d. Use a graphing calculator or computer to graph the Temperature function over the same time interval as in the experiment and visually check how well the graph of the function *T* fits the data points gathered from the experiment.